

# BLOCK-TOEPLITZ DETERMINANTS, CHESS TABLEAUX, AND THE TYPE $\widehat{A}_1$ GEISS-LECLERC-SCHRÖER $\varphi$ -MAP

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**ABSTRACT.** We evaluate the Geiss-Leclerc-Schröer  $\varphi$ -map for *shape modules* over the preprojective algebra  $\Lambda$  of type  $\widehat{A}_1$  in terms of matrix minors arising from the block-Toeplitz representation of the loop group  $\mathrm{SL}_2(\mathcal{L})$ . Conjecturally these minors are among the cluster variables for coordinate rings of unipotent cells within  $\mathrm{SL}_2(\mathcal{L})$ . In so doing we compute the Euler characteristic of any *generalized flag variety* attached to a shape module by counting standard tableaux of requisite shape and *parity*; alternatively by counting *chess tableaux* of requisite shape and content.

## INTRODUCTION:

In [10] C. Geiss, B. Leclerc, and J. Schröer initiated the study of the (generalized) tilting theory for preprojective algebras of Dynkin type  $\Delta$  vis á vie the cluster algebra structure of the coordinate ring of the maximal unipotent group  $U_+$  attached to  $\Delta$  under the Cartan-Killing classification; see [1], [4], and [5] regarding *cluster algebras*. Specifically they construct an explicit map  $\varphi$  from the module category of the preprojective algebra to the coordinate ring of the corresponding maximal unipotent group which transforms exceptional objects into cluster variables and maximal rigid modules into clusters. The  $\varphi$ -map can be interpreted (and this is the view initially taken here) as type of partition function which records the Euler characteristics of *generalized flag varieties* attached to the module.

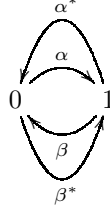
Recently both [12] and [2] have independently made the new step of examining the GLS  $\varphi$ -map in the affine setting. In particular [2] studied examples of unipotent cells of the loop group  $\mathrm{SL}_2(\mathcal{L})$  and proved that their coordinate rings — in accordance with the predictions made in [1] — are cluster algebras of geometric type. This was accomplished in part by evaluating the type  $\widehat{A}_1$  GLS  $\varphi$ -map for a fixed family of nilpotent finite dimensional modules over the preprojective algebra of type  $\widehat{A}_1$ . For each unipotent cell in  $\mathrm{SL}_2(\mathcal{L})$  the authors of [2] conjecture an explicit list of nilpotent  $\Lambda$ -modules whose images under the type  $\widehat{A}_1$  GLS  $\varphi$ -map form the initial seed generating the cluster algebra structure of the coordinate ring of the unipotent cell. Moreover conjecture (4.3) of [2] predicts determinantal expressions for these initial cluster variables.

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This paper evaluates the type  $\widehat{A}_1$  GLS  $\varphi$ -map over a class of nilpotent  $\Lambda$ -modules called *shape modules* — indeed a class which properly contains those modules stipulated in conjectures (4.1)-(4.3) of [2] — and expresses the result determinantly in order to settle (4.3) of [2]. The proof entails computing the Euler characteristic of any generalized flag variety attached to a shape module; this is accomplished combinatorially by counting standard tableaux of requisite shape and parity. We now go into more detail:

Recall that the preprojective algebra  $\Lambda$  of type  $\widehat{A}_1$  is defined as the quotient of the *path algebra* associated to the quiver  $Q$



by the ideal  $I$  generated by  $\alpha^*\alpha - \beta\beta^*$  and  $\beta^*\beta - \alpha\alpha^*$ . Let  $e_0$  and  $e_1$  denote the idempotents of  $\Lambda$ .

For a finite dimensional left  $\Lambda$ -module  $M$  of dimension  $\dim M = n$  together with a choice of bit string  $\mathbf{d} = (d_1, \dots, d_n)$  in  $\{0, 1\}^n$  the *generalized flag variety*  $\mathcal{F}_{\mathbf{d}}^{\Lambda}(M)$  is the variety of all  $\Lambda$ -composition series  $M_n \supset \dots \supset M_0$  with  $M_n = M$  and  $M_0 = \{0\}$  such that  $M_t/M_{t-1} \simeq S_{d_t}$  whenever  $n \geq t \geq 1$ . Here  $S_0$  and  $S_1$  are the simple left  $\Lambda$ -modules associated to the vertices labeled 0 and 1 in the quiver  $Q$ .

**Definition 1** (Type  $\widehat{A}_1$  GLS  $\varphi$ -map). *Let  $\mathbf{i} = (i_1, \dots, i_k)$  be an alternating bit string in  $\{0, 1\}^k$  and let  $a_1, \dots, a_k$  denote — for the moment — formal variables. If  $M$  is a finite dimensional left  $\Lambda$ -module then*

$$\varphi_M(a_1, \dots, a_k) := \sum_{\mathbf{j} \in \mathbb{Z}_{\geq 0}^k} \chi\left(\mathcal{F}_{\mathbf{i}\mathbf{j}}^{\Lambda}(M)\right) \frac{a_1^{j_1} \cdots a_k^{j_k}}{j_1! \cdots j_k!}$$

where  $\mathbf{i}\mathbf{j}$  is the bit string in  $\{0, 1\}^n$  given by

$$(i_{\sigma(1)}, \dots, i_{\sigma(n)})$$

with  $n = j_1 + \dots + j_k$  and where

$$(1) \quad \sigma(t) := \min \{s \mid j_1 + \dots + j_s \geq t\}$$

whenever  $n \geq t \geq 1$ . The symbol  $\chi$  denotes Euler characteristic (for cohomology with compact support; see [8]).

Section (1) of this paper begins with a quick survey of partition and tableau combinatorics. The notion of *i-parity* of a standard tableau is defined together with the auxiliary notion of a *chess tableau*. Proposition (1) proves that the number of standard tableaux of shape  $\lambda$  and *i-parity*  $\mathbf{i}^{\mathbf{j}}$  equals  $j_1! \cdots j_k!$  times the number of chess tableaux of shape  $\lambda$  and content  $\mathbf{j}$  where  $\mathbf{i}$  is an alternating bit string in  $\{0, 1\}^k$  and  $\mathbf{j} \in \mathbb{Z}_{\geq 0}^k$ .

In section (2) we give a construction which associates to a pair  $\mu \subset \lambda$  of ordered partitions and choice of parity  $i = 0, 1$  a nilpotent left  $\Lambda$ -module called the *skew-shape module*. The modules considered in chapter (4) of [2] are particular examples. *Shape modules* are defined as skew-modules where the smaller partition  $\mu$  is empty. Following this section (3) details a proof of our first theorem:

**Theorem 1.** *Let  $M$  be a shape module of shape  $\lambda$ , parity  $i$ , and dimension  $n$ . If  $\mathbf{d} = (d_1, \dots, d_n)$  is a bit string in  $\{0, 1\}^n$  then the Euler characteristic  $\chi\left(\mathcal{F}_{\mathbf{d}}^{\Lambda}(M)\right)$  equals the number of standard tableaux  $T$  of shape  $\lambda$  whose *i-parity* equals  $\mathbf{d}$ .*

Conjecture (1) of section (3) refines Theorem (1) and tallies the number of  $\mathbb{F}_q$ -rational points of  $\mathcal{F}_{\mathbf{d}}^{\Lambda}(M)$  when  $\Lambda$  is viewed as an algebra over a finite field  $\mathbb{F}_q$  with  $q$  elements.

The *algebraic loop group*  $\mathrm{SL}_2(\mathcal{L})$  is the group consisting of all  $\mathcal{L}$ -valued  $2 \times 2$  matrices  $g = (g_{ij})$  with determinant 1 where  $\mathcal{L}$  is the Laurent polynomial ring  $\mathbb{C}[t, t^{-1}]$ . An element  $g \in \mathrm{SL}_2(\mathcal{L})$  is viewed as encoding a regular map  $g : \mathbb{C}^* \rightarrow \mathrm{SL}_2(\mathbb{C})$  given by  $z \mapsto (g_{ij}(z))$ ; a closed contour or “loop” in  $\mathrm{SL}_2(\mathbb{C})$  is obtained upon restricting the map to the circle group  $S^1$  in  $\mathbb{C}^*$  and taking its image — hence the name.

The *maximal unipotent subgroup*  $U_+$  is the subgroup of  $\mathrm{SL}_2(\mathcal{L})$  containing all loops  $g : \mathbb{C}^* \rightarrow \mathrm{SL}_2(\mathbb{C})$  which extend to 0 and for which  $g(0)$  is an upper triangular unipotent matrix. As  $\mathcal{L}$ -valued matrices

$$U_+ = \left\{ g \in \begin{pmatrix} 1 + t\mathbb{C}[t] & \mathbb{C}[t] \\ t\mathbb{C}[t] & 1 + t\mathbb{C}[t] \end{pmatrix} \mid \text{with } \det(g) = 1 \right\}.$$

For  $i = 0, 1$  and  $a \in \mathbb{C}^*$  let  $x_i : \mathbb{C}^* \rightarrow U_+$  denote the 1-parameter subgroups defined by

$$x_0(a) := \begin{pmatrix} 1 & 0 \\ at & 1 \end{pmatrix} \quad x_1(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

The set of elements  $\mathcal{O}_+$  which factorize as  $x_{i_1}(a_1) \cdots x_{i_k}(a_k)$  for some alternating bit string  $\mathbf{i} = (i_1, \dots, i_k)$  in  $\{0, 1\}^k$ , for some choice of parameters  $a_1, \dots, a_k$  in  $\mathbb{C}^*$ , and for some  $k$  is a Zariski open subset within  $U_+$ . Consequently a regular function over  $U_+$  is uniquely determined by its values over  $\mathcal{O}_+$ .

In section (4) we define regular functions  $\Delta_{\mu, \lambda}^{(i)} : \mathrm{SL}_2(\mathcal{L}) \rightarrow \mathbb{C}$  indexed by a choice of parity  $i = 0, 1$  together with a pair of partitions  $\mu$  and  $\lambda$ . These functions are expressed as minors of the infinite block-Toeplitz matrix  $T_g$  associated to the argument  $g \in \mathrm{SL}_2(\mathcal{L})$ . In particular the minors considered in [2] are of the form  $\Delta_{\mu, \lambda}^{(i)}$ .

These minors are shown to satisfy an  $i$ -Pieri rule — reminiscent of the generalized Pieri identities considered by [14] — and as such behave like Schur polynomials which carry a parity. This view is reinforced by remark (7) in section (5) which expresses  $\Delta_{\emptyset, \lambda}^{(i)}$  as a generating function for chess tableaux of shape  $\lambda$  and parity  $i$ . Section (4) ends with conjecture (2) which claims that the type  $\widehat{A_1}$ -generalized minors of Fomin-Zelevinsky (see [6]) are among these block-Toeplitz matrix minors.

In section (5) the restriction of the minors  $\Delta_{\mu, \lambda}^{(i)}$  to the maximal unipotent subgroup  $U_+$  of the loop group are studied combinatorially by means of pairwise non-crossing families of paths in a weighted planar network  $\Gamma_1(\mathbf{a})$ . Proposition (3) sets up a weight and content preserving bijection between families of non-crossing paths and chess tableaux which is then used to prove the main result of this paper:

**Theorem 2.** *Let  $M$  be a shape module of shape  $\lambda$ , parity  $i$ , and dimension  $n$ . Let  $\mathbf{i} = (i_1, \dots, i_k)$  be an alternating bit string in  $\{0, 1\}^k$  and let  $a_1, \dots, a_k$  be parameters in  $\mathbb{C}^*$  then*

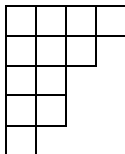
$$\Delta_{\emptyset, \lambda}^{(i)}(x_{i_1}(a_1) \cdots x_{i_k}(a_k)) = \varphi_M(a_1, \dots, a_k)$$

## 1. PARTITIONS AND TABLEAUX:

By a partition  $\lambda$  we will mean a non-increasing infinite sequence of non-negative integers  $(\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \dots)$  with is eventually zero; as short hand we only write the non-zero terms. The size  $|\lambda|$  of  $\lambda$  is defined as the sum  $\lambda_0 + \lambda_1 + \dots$  and we say the  $\lambda$  is a partition of  $m$  if  $|\lambda| = m$ . In the context of partitions the symbol  $\emptyset$  will denote the *empty partition* defined by  $\lambda_n = 0$  whenever  $n \geq 0$ . For a partition  $\lambda$  let  $N_\lambda := \max \{n \mid \lambda_n > 0\}$  with the understanding that  $N_\emptyset = 0$ . Given a partition  $\lambda$ , a choice of parity  $i = 0, 1$ , and a non-negative integer  $N$  define the  $\text{set}_N^{(i)}(\lambda)$  by

$$\text{set}_N^{(i)}(\lambda) := \{\lambda_n + i - n \mid N \geq n \geq 0\}$$

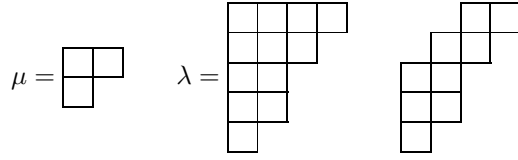
In this paper a partition  $\lambda$  of  $n$  will be synonymous with its Young diagram or *shape*: This is a left-justified arrangement of  $n$  boxes into rows whereby the top row contains  $\lambda_0$  boxes, the next  $\lambda_1$ , and so on. For instance shape of the partition  $\lambda = (4, 3, 2, 2, 1)$  is



Each box in a shape will be coordinatized by its row and column positions  $s, t$  measured from the top and left respectively. We employ the convention that the top row and left-most column are counted as row and column zero.

**Definition 2.** Let  $\lambda$  be a partition and let  $i = 0, 1$  be a choice of parity. The  $i$ -**parity** of a box with row and column coordinates  $s, t$  in the shape of  $\lambda$  is by definition the parity of  $s + t + i$ .

By definition two partitions  $\mu = (\mu_0 \geq \mu_1 \geq \dots)$  and  $\lambda = (\lambda_0 \geq \lambda_1 \geq \dots)$  are ordered  $\mu \subseteq \lambda$  if  $\mu_t \leq \lambda_t$  for all integers  $t \geq 0$ . The symbol  $\subseteq$  is justified by the fact that  $\mu \subseteq \lambda$  if and only if the shape of  $\mu$  is contained in the shape of  $\lambda$ . In this case let  $\lambda/\mu$  denote the *skew-shape* obtained by removing  $\mu$  from  $\lambda$  as shapes. For example: If  $\mu = (2, 1)$  and  $\lambda = (4, 3, 2, 2, 1)$  then  $\mu$ ,  $\lambda$ , and  $\lambda/\mu$  are depicted respectively by



Let  $\lambda$  be a partition and  $\mathbf{q} = (q_1, \dots, q_m)$  be a non-negative integer  $m$ -tuple in  $\mathbb{Z}_{\geq 0}^m$  such that  $q_1 + \dots + q_m = |\lambda|$ . For the purposes of this paper, a *semi-standard tableau*  $T$  of shape  $\lambda$  and content  $\mathbf{q}$  is an assignment whereby each box in shape of  $\lambda$  is labeled by an integer  $t$  in  $[1 \dots m]$  such that

- each index  $t \in [1 \dots m]$  is used exactly  $q_t$  times
- the indices  $t$  strictly increase when read from left to right in each row
- the indices  $t$  weakly increase when read from top to bottom in each column

In the event  $n \geq |\lambda|$  and  $\mathbf{q} = (q_1, \dots, q_n)$  is a bit string in  $\{0, 1\}^n$  with  $q_1 + \dots + q_n = |\lambda|$  we say that  $T$  is a *standard tableau of content  $\mathbf{q}$* . Note that if in this case the indices  $t$  of  $T$  are read from top to bottom in each column they will necessarily strictly increase. By convention a standard tableau  $T$  of shape  $\lambda$  without any reference to content will be understood to be a standard tableau of shape  $\lambda$  and content  $\mathbf{q} \in \{0, 1\}^{|\lambda|}$  where  $q_t = 1$  whenever  $|\lambda| \geq t \geq 1$ . Below are examples of a standard and a semi-standard tableaux of shape  $\lambda = (4, 3, 2, 2, 1)$  with the later having content  $\mathbf{q} = (2, 1, 1, 3, 2, 2, 1)$ :

1	2	5	7				
3	6	9					
4	10						
8	12						
11							

1	2	4	5				
1	3	4					
4	6						
5	6						
7							

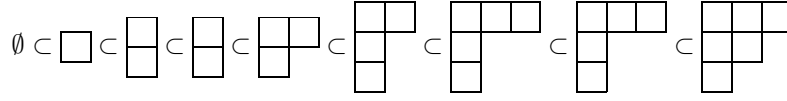
For a partition  $\lambda$  and a bit string  $\mathbf{q}$  in  $\{0, 1\}^n$  with  $n \geq |\lambda|$  and  $q_1 + \dots + q_n = |\lambda|$  the set of all standard tableaux of shape  $\lambda$  and content  $\mathbf{q}$  will be denoted  $\text{Tab}_{\mathbf{q}}(\lambda)$ ; the set of all standard tableaux of shape  $\lambda$  will be denoted  $\text{Tab}(\lambda)$

Given an arbitrary non-negative integer  $m$ -tuple  $\mathbf{q} = (q_1, \dots, q_m)$  in  $\mathbb{Z}_{\geq 0}^m$  let  $[\mathbf{q}]$  denote the *indicator set*, i.e.

$$[\mathbf{q}] := \{t \mid q_t \neq 0\}$$

For  $n \geq m$  let  $\{0, 1\}_m^n$  denote the set of bit strings  $\mathbf{q}$  such that  $[\mathbf{q}]$  has cardinality  $m$ . We may express the indicator set of a bit string  $\mathbf{q}$  in  $\{0, 1\}_m^n$  as  $[\mathbf{q}] = \{r_1 < \dots < r_m\}$ . For a partition  $\lambda$  of  $m$  and a tableau  $T$  in  $\text{Tab}_{\mathbf{q}}(\lambda)$  let  $\overline{T}$  be the tableau in  $\text{Tab}(\lambda)$  obtained by replacing each  $r_t$  in  $T$  by  $t$  whenever  $m \geq t \geq 1$ . Clearly the mapping  $T \mapsto \overline{T}$  defines a bijection between  $\text{Tab}_{\mathbf{q}}(\lambda)$  and  $\text{Tab}(\lambda)$ .

Assume  $\lambda$  is a partition of  $m$  and fix a bit string  $\mathbf{q} \in \{0, 1\}_m^n$  and express its indicator set as  $[\mathbf{q}] = \{r_1 < \dots < r_m\}$ . A  **$\mathbf{q}$ -step flag of partitions** in  $\lambda$  is a weakly increasing sequence of partitions  $\lambda^{(0)} \subseteq \dots \subseteq \lambda^{(n)}$  with  $\lambda^{(0)} = \emptyset$  and  $\lambda^{(n)} = \lambda$  such that the shapes of  $\lambda^{(t-1)}$  and  $\lambda^{(t)}$  differ by exactly one corner box whenever  $q_t = 1$  and are equal whenever  $q_t = 0$ . Associate to a  **$\mathbf{q}$ -step flag of partitions**  $(\lambda^{(t)})_{n \geq t \geq 0}$  a standard tableau  $T$  with content  $\mathbf{q}$  obtained by labeling the box of  $\lambda$  which is deleted from  $\lambda^{(t)}$  in order to get  $\lambda^{(t-1)}$  by  $r_t$ . For example when  $\lambda = (3, 2, 1)$  and  $\mathbf{q} = (1, 1, 0, 1, 1, 1, 0, 1)$  the  **$\mathbf{q}$ -step flag of partitions** (depicted here in terms of shapes)



corresponds to



The correspondence between  **$\mathbf{q}$ -step flags of partitions** in  $\lambda$  and standard tableaux of shape  $\lambda$  and content  $\mathbf{q}$  is clearly bijective.

**Definition 3.** Let  $\lambda$  be a partition of  $n$  and let  $i = 0, 1$  be a choice of parity. The  **$i$ -parity** of a standard tableau  $T$  of shape  $\lambda$  is by definition the bit string  $\mathbf{d}_T = (d_1, \dots, d_n)$  in  $\{0, 1\}^n$  where  $d_t$  equals the  $i$ -parity of the box labeled by  $t$  in  $T$ . Let  $\text{Tab}^{(i)}(\lambda; \mathbf{d})$  denote the set of all such tableaux.

**Definition 4.** Let  $\lambda$  be a partition, let  $i = 0, 1$  be a choice of parity, and let  $\mathbf{j} = (j_1, \dots, j_k)$  be a non-negative integer  $k$ -tuple in  $\mathbb{Z}_{\geq 0}^k$  such that  $j_1 + \dots + j_k = |\lambda|$ . A **chess tableau**  $T$  of shape  $\lambda$ , parity  $i$ , and content  $\mathbf{j}$  is a semi-standard tableau of shape  $\lambda$  and content  $\mathbf{j}$  with the added constraint that the  $i$ -parity of each box in  $T$  equals the parity of the index  $t$  labeling that box. Let  $\text{Chess}_{\mathbf{j}}^{(i)}(\lambda)$  denote the set of all chess tableaux of shape  $\lambda$ , parity  $i$ , and content  $\mathbf{j}$ . Let  $\text{Chess}^{(i)}(\lambda)$  denote the disjoint union

$$\text{Chess}^{(i)}(\lambda) := \bigsqcup_{\mathbf{j} \in \mathbb{Z}_{\geq 0}^k} \text{Chess}_{\mathbf{j}}^{(i)}(\lambda)$$

**Remark 1.** The parity condition for chess tableaux forces the row and column entries to increase strictly.

Chess tableaux seem to have been considered first by Chow, Erisson, and Fan in [3]. Among other things, they independently interpret chess tableaux in terms of *rat races* which are essentially the non-crossing path configurations which we consider in section 5.

**Proposition 1.** *Let  $\lambda$  be a partition of  $n$ , let  $i = 0, 1$  be a choice of parity, let  $\mathbf{d}$  be a bit string in  $\{0, 1\}^n$ , let  $\mathbf{i} = (i_1, \dots, i_n)$  be an alternating bit string in  $\{0, 1\}^n$ , and let  $\mathbf{j} = (j_1, \dots, j_k)$  be a non-negative integer  $k$ -tuple in  $\mathbb{Z}_{\geq 0}^k$  such that  $\mathbf{i}^{\mathbf{j}} = \mathbf{d}$  then*

$$\left| \text{Tab}^{(i)}(\lambda; \mathbf{d}) \right| = j_1! \cdots j_k! \left| \text{Chess}_{\mathbf{j}}^{(i^*)}(\lambda) \right|$$

where  $i^*$  is the parity of  $i + i_1 + 1$ .

*Proof.* Recall the definition of  $\sigma(t)$  given in equation (1) of the introduction. Note that  $\sigma(t) \in [j]$  whenever  $n \geq t \geq 1$ . Clearly  $\sigma(t) \geq \sigma(t')$  whenever  $t > t'$ . If  $T$  be a tableau in  $\text{Tab}^{(i)}(\lambda; \mathbf{d})$  then the filling  $\Sigma(T)$  of  $\lambda$  obtained by replacing each index  $t$  in  $T$  by  $\sigma(t)$  is weakly increasing in both rows and columns. The parity of  $\sigma(t)$  equals the parity of  $d_t + i_1 + 1$  whenever  $n \geq t \geq 1$ . Consequently the  $i$ -parity of the box labeled  $t$  in  $T$  equals  $d_t$  if and only if the  $i^*$ -parity of the box labeled  $t$  in  $T$  equals the parity of  $\sigma(t)$ . By assumption the  $i$ -parity of the box in  $T$  labeled  $t$  equals  $d_t$  whenever  $n \geq t \geq 1$  therefore the  $i^*$ -parity of the box labeled  $\sigma(t)$  in  $\Sigma(T)$  equals the parity of  $\sigma(t)$ ; hence  $\Sigma(T) \in \text{Chess}^{(i^*)}(\lambda)$ . Moreover for each  $s \in [1 \dots k]$  the cardinality of  $\sigma^{-1}(s)$  is exactly  $j_s$  and so  $\Sigma(T) \in \text{Chess}_{\mathbf{j}}^{(i^*)}(\lambda)$ .

If  $S$  is a chess tableau in  $\text{Chess}_{\mathbf{j}}^{(i^*)}(\lambda)$  let  $T$  be a filling of  $\lambda$  obtained by replacing each label  $s$  occurring in  $S$  by some choice of element  $t$  in  $\sigma^{-1}(s)$  with the rule that pre-images in  $\sigma^{-1}(s)$  are not reused once they are selected. As indicated in the remark above, any chess tableau is in fact row and column strict and since  $\sigma(t) \geq \sigma(t')$  whenever  $t > t'$  it follows that  $T$  will be a standard tableau. Moreover the  $i$ -parity of the box labeled  $t$  will be  $d_t$ . Consequently the mapping  $\Sigma : \text{Tab}^{(i)}(\lambda; \mathbf{d}) \longrightarrow \text{Chess}_{\mathbf{j}}^{(i^*)}(\lambda)$  is surjective.

Since the construction described in the surjectivity argument prescribes that each index in  $s \in [j]$  is chosen once and since there are  $j_s$ -choices to be made it follows that there are exactly  $j_s!$  possibilities for each  $s$ . The proposition now follows.  $\square$

For a standard tableau  $T$  of shape  $\lambda$  and an index  $t$  let  $\text{row}_t$  and  $\text{col}_t$  respectively denote the row and column positions of the box labeled by  $t$  in  $T$

**Definition 5.** *Let  $\lambda$  be a partition of  $n$  and let  $\mathbf{d} = (d_1, \dots, d_n)$  be a bit string in  $\{0, 1\}^n$ . A pair of indices  $\{s, t\}$  is called a **d-transposition pair** for a standard tableau  $T$  of shape  $\lambda$  if  $d_s = d_t$  and the tableau  $T'$  obtained by exchanging the positions of  $s$  and  $t$  remains standard. A **d-transposition pair**  $\{s, t\}$  is said to be **grounded** if  $s < t$  and  $\text{row}_s < \text{row}_t$  and  $\text{col}_t < \text{col}_s$ . Define the **ground state**  $\text{gr}(T)$  of a standard tableau  $T$  as the total number of grounded **d-transposition pairs** of  $T$ .*

2. SHAPE MODULES OVER  $\Lambda$ :

We may always regard a finite dimensional left  $\Lambda$ -module  $M$  as a module over the polynomial ring  $\mathbb{C}[\delta]$  whereby  $\delta := \alpha + \beta$ . If in addition  $M$  is nilpotent as a  $\Lambda$ -module then its isomorphism type when viewed as a  $\mathbb{C}[\delta]$ -module is determined by the partition  $\lambda$  of  $\dim M$  which encodes the Jordan type of  $\delta = \alpha + \beta$  considered as a nilpotent endomorphism on  $M$ ; we call  $\lambda$  the  $\mathbb{C}[\delta]$ -*partition type* of  $M$ . In this case we can visualize the  $\mathbb{C}[\delta]$ -module structure on  $M$  using the shape of  $\lambda$ : Each box in the shape corresponds to a basis vector in  $M$  and each row corresponds to an indecomposable  $\mathbb{C}[\delta]$ -summand of  $M$  — with the convention that the action of  $\delta$  is depicted as going from right to left in each row. For example the shape and corresponding  $\mathbb{C}[\delta]$ -module associated to the partition  $\lambda = (4, 2)$  are

$$\lambda = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad \begin{array}{l} v_{0,0} \xleftarrow{\delta} v_{0,1} \xleftarrow{\delta} v_{0,2} \xleftarrow{\delta} v_{0,3} \\ \\ v_{1,0} \xleftarrow{\delta} v_{1,1} \end{array}$$

where  $v_{s,t}$  is the basis vector corresponding to the box in located in row  $s$  and column  $t$  of the shape of  $\lambda$  — with the proviso that the top row and left most column of  $\lambda$  both have coordinates equal to 0.

It is well known (e.g. see [16] chapter II section 2) that if  $M$  is a nilpotent  $\mathbb{C}[\delta]$ -module and  $N$  is a proper  $\mathbb{C}[\delta]$ -submodule then  $\mu \subset \lambda$  where  $\mu$  and  $\lambda$  are the  $\mathbb{C}[\delta]$ -partition types of  $N$  and  $M$  respectively.

For a finite dimensional left  $\Lambda$ -module  $M$  of dimension  $\dim M = n$  let  $\mathcal{F}^\delta(M)$  denote the variety of all  $\mathbb{C}[\delta]$ -composition series, i.e. complete flags  $M_n \supset \cdots \supset M_0$  with  $M_n = M$  and  $M_0 = \{0\}$  such that  $M_{t-1}$  is a maximal  $\mathbb{C}[\delta]$ -submodule of  $M_t$  whenever  $n \geq t \geq 1$ .

If  $M$  is a nilpotent  $\mathbb{C}[\delta]$ -module of dimension  $\dim M = n$  and  $\mathbb{C}[\delta]$ -partition type  $\lambda$  then any  $\mathbb{C}[\delta]$ -composition series  $M_n \supset \cdots \supset M_0$  in  $M$  gives rise by the remarks made earlier to a flag of partitions  $\lambda^{(n)} \supset \cdots \supset \lambda^{(0)}$  in  $\lambda$  where  $\lambda^{(t)}$  is the  $\mathbb{C}[\delta]$ -partition type of  $M_t$ . The standard tableau  $T$  of shape  $\lambda$  which records this flag of partitions will be called the  $\mathbb{C}[\delta]$ -*tableau type* of the  $\mathbb{C}[\delta]$ -composition series.

**Definition 6.** *Let  $M$  be a nilpotent left  $\Lambda$ -module of dimension  $\dim M = n$  and  $\mathbb{C}[\delta]$ -partition type  $\lambda$ . For a standard tableau  $T$  of shape  $\lambda$  let  $\Omega_T^\delta(M)$  denote the collection of all  $\mathbb{C}[\delta]$ -composition series of  $\mathbb{C}[\delta]$ -tableau type  $T$  in  $\mathcal{F}^\delta(M)$ . Set  $\Omega_{T,d}^\Lambda(M) := \Omega_T^\delta(M) \cap \mathcal{F}_d^\Lambda(M)$  for  $d \in \{0, 1\}^n$ .*

Macdonald relates in [16] that Spaltenstein proved in [18] that  $\Omega_T^\delta(M)$  is a smooth irreducible locally closed subvariety of  $\mathcal{F}^\delta(M)$ ; moreover  $\Omega_T^\delta(M)$  is a disjoint union of subvarieties each of which is isomorphic to an affine space.

**Lemma 1.**

$$\chi\left(\mathcal{F}_d^\Lambda(M)\right) = \sum_{T \in \text{Tab}(\lambda)} \chi\left(\Omega_{T,d}^\Lambda(M)\right)$$



*Proof.* The subset  $\Omega_{T,d}^\Lambda(M)$  is a locally closed subvariety of  $\mathcal{F}_d^\Lambda(M)$  owing to the fact that  $\Omega_T^\delta(M)$  is locally closed in  $\mathcal{F}^\delta(M)$ . The lemma must hold since Euler characteristic  $\chi$  (for cohomology with compact support; see [8]) is additive over disjoint unions of locally closed subvarieties and because

$$\mathcal{F}_d^\Lambda(M) = \bigsqcup_{T \in \text{Tab}(\lambda)} \Omega_{T,d}^\Lambda(M)$$

□

**Definition 7.** Let  $i = 0, 1$  be a choice of parity and let  $\lambda = (\lambda_0 \geq \lambda_1 \geq \dots)$  and  $\mu = (\mu_0 \geq \mu_1 \geq \dots)$  be two partitions such that  $\mu \subset \lambda$ . The **skew-shape module**  $M$  of **skew-shape**  $\lambda/\mu$  and **parity**  $i$  is the nilpotent left  $\Lambda$ -module of dimension  $|\lambda| - |\mu|$  with basis

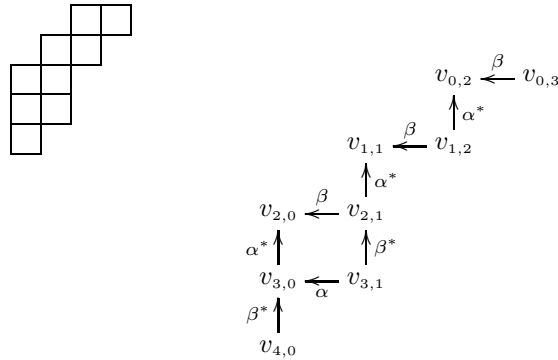
$$\left\{ v_{s,t} \mid \mu_s \leq t < \lambda_s \right\}$$

for which the  $\alpha$ ,  $\beta$ ,  $\alpha^*$ , and  $\beta^*$  actions are given by

$$\begin{aligned} \alpha v_{s,t} &= \begin{cases} v_{s-1,t} & s+t+i \text{ even} \\ 0 & \text{otherwise} \end{cases} & \alpha^* v_{s,t} &= \begin{cases} v_{s,t-1} & s+t+i \text{ odd} \\ 0 & \text{otherwise} \end{cases} \\ \beta v_{s,t} &= \begin{cases} v_{s-1,t} & s+t+i \text{ odd} \\ 0 & \text{otherwise} \end{cases} & \beta^* v_{s,t} &= \begin{cases} v_{s,t-1} & s+t+i \text{ even} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and zero otherwise. If  $\mu = \emptyset$  we call  $M$  instead a **shape module** of shape  $\lambda$  and parity  $i$ . In addition the shape module associated to  $\lambda = \emptyset$  is by definition the zero module.

This module can be depicted by a grid of north and east pointing arrows, each representing the action of either  $\alpha$ ,  $\beta$ ,  $\alpha^*$ , and  $\beta^*$ , which is subordinate to the skew-shape  $\lambda/\mu$ . For instance the case of  $i = 0$ ,  $\mu = (2, 1)$ , and  $\lambda = (4, 3, 2, 2, 1)$  is shown here together with the corresponding skew-shape:



**Remark 2.** The preprojective relations  $\alpha^* \alpha - \beta \beta^* = 0$  and  $\beta^* \beta - \alpha \alpha^* = 0$  are clearly valid for any skew-shape module and any parity. Clearly the  $\mathbb{C}[\delta]$ -partition type of the shape module  $M$  of shape  $\lambda$  of any parity  $i$  is  $\lambda$ .

## 3. EVALUATION OF EULER CHARACTERISTICS:

The techniques used in this section are adopted from those described by Geiss-Leclerc-Schröer in sections 2 and 3 of their publication [11]. In addition we have tried to reconcile our notation with theirs as much as possible.

Let  $M$  be a shape module of shape  $\lambda$ , parity  $i$ , and dimension  $\dim M = |\lambda| = n$ . Assume also that  $\lambda$  has  $k$  parts. Let  $P$  denote the  $\Lambda$ -submodule of  $M$  spanned by the vectors  $\{v_{t,0} \mid 0 \leq t < k\}$  associated to the first column of  $\lambda$ . Let  $Q$  denote the span of the remaining basis vectors  $\{v_{s,t} \mid 1 \leq t < \lambda_s\}$ . Evidently both  $P$  and the quotient module  $M/P$  are shape modules of parity  $i$  and  $1-i$  and shape  $1^k$  and  $\bar{\lambda}$  respectively where  $\bar{\lambda}_t := \lambda_t - 1$  provided  $\lambda_t > 0$  and zero otherwise. Note  $Q$  is a  $\mathcal{A}$ -submodule of  $M$  where  $\mathcal{A}$  is the  $\mathbb{C}$ -subalgebra of  $\Lambda$  generated by  $\alpha^*$ ,  $\beta^*$ ,  $e_0$ , and  $e_1$ ; moreover  $Q$  and  $M/P$  are isomorphic when regarded as  $\mathcal{A}$ -modules. Let  $q : M \rightarrow M/P$  and  $\text{pr}_Q : M \rightarrow Q$  denote the quotient and projection maps respectively.

Let  $\mathbf{c}$  be a bit string in  $\{0,1\}_k^n$ , let  $\bar{\mathbf{c}}$  denote the bit string  $(1-c_1, \dots, 1-c_n)$  in  $\{0,1\}_{n-k}^n$ , and let  $\{\mathbf{d}\}$  be an arbitrary bit string in  $\{0,1\}^n$ . Let  $\mathcal{F}_d^\Lambda(M; \mathbf{c})$  denote the subset of  $\mathcal{F}_d^\Lambda(M)$  consisting of all  $\Lambda$ -composition series  $(M_t)_{n \geq t \geq 0}$  such that

$$\dim \frac{M_t \cap P}{M_{t-1} \cap P} = c_t$$

whenever  $n \geq t \geq 1$ . Clearly this condition is equivalent to

$$\dim \frac{\text{pr}_Q M_t}{\text{pr}_Q M_{i-t}} = 1 - c_t$$

whenever  $n \geq t \geq 1$ . An easy application of the Zassenhaus butterfly lemma for vector spaces shows that

$$\mathcal{F}_d^\Lambda(M) = \bigsqcup_{\mathbf{c} \in \{0,1\}_k^n} \mathcal{F}_d^\Lambda(M; \mathbf{c})$$

**Lemma 2.** *Let  $M$  be a finite dimensional nilpotent left  $\Lambda$ -module of  $\mathbb{C}[\delta]$ -partition type  $\lambda$ . For bit strings  $\mathbf{c} \in \{0,1\}_k^n$  and  $\mathbf{d} \in \{0,1\}^n$*

$$\mathcal{F}_d^\Lambda(M; \mathbf{c}) = \bigsqcup_T \Omega_{T,d}^\Lambda(M)$$

where the union is taken over all standard tableaux  $T$  of shape  $\lambda$  whose first column content is  $[\mathbf{c}]$ .

*Proof.* Note first that if  $(M_t)_{n \geq t \geq 0}$  is a  $\Lambda$ -composition series in  $M$  then  $\dim M_t \cap P$  equals the number of parts of the partition  $\lambda^{(t)}$  associated to  $M_t$  and hence equal to the number of rows in the shape of  $\lambda^{(t)}$ . Recall that box is labeled  $t$  in the standard tableau  $T$  associated to  $(M_t)_{n \geq t \geq 0}$  if and only if the shape of  $\lambda^{(t)}$  contains the box while the shape of  $\lambda^{(t-1)}$  does not. Consequently a box in the first column of  $T$  is labeled  $t$  if and only if the number of rows in the shape of  $\lambda^{(t)}$  is one more than the

number of rows in the shape of  $\lambda^{(t-1)}$ . Equivalently a box in the first column of  $T$  is labeled  $t$  if and only if  $\dim M_t \cap P = 1 + \dim M_{t-1} \cap P$ . Therefor  $t$  labels a box in the first column of  $T$  if and only if  $c_t = 1$ .  $\square$

It follows from lemma (2) that each  $\mathcal{F}_d^\Lambda(M; \mathbf{c})$  is a locally closed subvariety of  $\mathcal{F}_d^\Lambda(M)$ . For bit strings  $\mathbf{c} \in \{0, 1\}_k^n$  and  $\mathbf{d} \in \{0, 1\}^n$  let  $\pi_c : \mathcal{F}_d^\Lambda(M; \mathbf{c}) \longrightarrow \mathcal{F}_{c;d}^\Lambda(P) \times \mathcal{F}_{\bar{c};d}^\Lambda(Q)$  be the map given by

$$(M_t)_{n \geq t \geq 0} \mapsto (M_t \cap P)_{n \geq t \geq 0} \times (\text{pr}_Q(M_t))_{n \geq t \geq 0}$$

where  $\mathcal{F}_{c;d}^\Lambda(P)$  denotes the variety of all  $\mathbf{c}$ -step  $\Lambda$ -composition series, i.e.  $P_n := P \supseteq \cdots \supseteq P_0 := \{0\}$  in  $P$  such that

$$\frac{P_t}{P_{t-1}} \simeq \begin{cases} S_{d_t} & \text{if } c_t = 1 \\ \{0\} & \text{if } c_t = 0 \end{cases}$$

whenever  $n \geq t \geq 1$ . Bearing a slight abuse of terminology and notation  $\mathcal{F}_{\bar{c};d}^\Lambda(Q)$  denotes the variety of all  $\bar{\mathbf{c}}$ -step flags  $Q_n := Q \supseteq \cdots \supseteq Q_0 := \{0\}$  in  $Q$  such that  $q(Q_t)$  is a  $\Lambda$ -submodule of  $M/P$  and

$$\frac{q(Q_t)}{q(Q_{t-1})} \simeq \begin{cases} S_{d_t} & \text{if } c_t = 0 \\ \{0\} & \text{if } c_t = 1 \end{cases}$$

whenever  $n \geq t \geq 1$ .

Express the indicator sets of  $\mathbf{c}$  and  $\bar{\mathbf{c}}$  as  $[\mathbf{c}] = \{r_1 < \cdots < r_k\}$  and  $[\bar{\mathbf{c}}] = \{s_1 < \cdots < s_{n-k}\}$  respectively; also set  $r_0 = s_0 = 0$ . Set  $\mathbf{e} = (e_1, \dots, e_k)$  with  $e_t = d_{r_t}$  and  $\mathbf{f} = (f_1, \dots, f_{n-k})$  with  $f_t = d_{s_t}$ . Clearly  $\mathcal{F}_{c;d}^\Lambda(P)$  and  $\mathcal{F}_{\mathbf{e}}^\Lambda(P)$  are isomorphic as varieties, and so are  $\mathcal{F}_{\bar{c};d}^\Lambda(Q)$  and  $\mathcal{F}_{\mathbf{f}}^\Lambda(M/P)$ . Indeed the isomorphisms  $\psi_P$  and  $\psi_Q$  in each case are given by

$$\begin{aligned} \mathcal{F}_{c;d}^\Lambda(P) \ni (P_t)_{n \geq t \geq 0} &\xrightarrow{\psi_P} (P_{r_t})_{k \geq t \geq 0} \in \mathcal{F}_{\mathbf{e}}^\Lambda(P) \\ \mathcal{F}_{\bar{c};d}^\Lambda(Q) \ni (Q_t)_{n \geq t \geq 0} &\xrightarrow{\psi_Q} (q(Q_{s_t}))_{n-k \geq t \geq 0} \in \mathcal{F}_{\mathbf{f}}^\Lambda(M/P) \end{aligned}$$

Any  $\mathbf{c}$ -step composition series  $(P_i)_{n \geq i \geq 0}$  in  $\mathcal{F}_{c;d}^\Lambda(P)$  determines a  $\mathbf{c}$ -step flag of partitions  $1^{d_n} \supseteq \cdots \supseteq 1^{d_0}$  where  $d_t = c_1 + \cdots + c_t$  and  $d_0 = 0$ . This in turn corresponds to a standard tableau  $R$  of shape  $1^k$  of content  $\mathbf{c}$ . Let  $\Omega_{R,d}^\Lambda(P)$  denote the set of all  $\mathbf{c}$ -step composition series in  $\mathcal{F}_{c;d}^\Lambda(P)$  associated to the standard tableau  $R \in \text{Tab}_{\mathbf{c}}(1^k)$ .

Similarly any  $\bar{\mathbf{c}}$ -step composition series  $(Q_t)_{n \geq t \geq 0}$  in  $\mathcal{F}_{\bar{c};d}^\Lambda(Q)$  determines a  $\bar{\mathbf{c}}$ -step flag of partitions  $\bar{\lambda}^{(n)} \supseteq \cdots \supseteq \bar{\lambda}^{(0)}$  in  $\bar{\lambda}$  where  $\bar{\lambda}^{(t)}$  is the  $\mathbb{C}[\delta]$ -partition type of the  $\Lambda$ -submodule  $q(Q_t)$  of  $M/P$ . Let  $S$  be the standard tableau of shape  $\bar{\lambda}$  and

content  $\bar{\mathbf{c}}$  corresponding to this  $\bar{\mathbf{c}}$ -step flag. Let  $\Omega_{S,d}^\Lambda(Q)$  denote the set of  $\bar{\mathbf{c}}$ -step composition series in  $\mathcal{F}_{1-\mathbf{c},d}^\Lambda(Q)$  associated to the standard tableau  $S \in \text{Tab}_{\bar{\mathbf{c}}}(\bar{\lambda})$ .

Clearly  $\psi_P(\Omega_{R,d}^\Lambda(P)) = \Omega_{\bar{R},e}^\Lambda(P)$  and  $\psi_Q(\Omega_{S,d}^\Lambda(Q)) = \Omega_{\bar{S},f}^\Lambda(M/P)$  where  $\bar{R}$  and  $\bar{S}$  are standard tableaux obtained from  $R$  and  $S$  by replacing each index  $r_t$  in  $R$  by  $t$  and each index  $s_t$  in  $S$  by  $t$  respectively.

**Proposition 2.** *Let  $M$  be a shape module of shape  $\lambda$  and parity  $i$ . Let  $T$  be a standard tableau of shape  $\lambda$  whose left-most column is labeled by indices in  $[\mathbf{c}]$ . Let  $R \in \text{Tab}_{\mathbf{c}}(1^k)$  and  $S \in \text{Tab}_{\bar{\mathbf{c}}}(\bar{\lambda})$  be the pair of standard tableaux respectively obtained by taking the 1-st column and its complement in  $T$ , then*

$$\pi_{\mathbf{c}}(\Omega_{T,d}^\Lambda(M)) = \Omega_{R,d}^\Lambda(P) \times \Omega_{S,d}^\Lambda(Q)$$

*Proof.* Its is an immediate consequence of the definition of  $\Omega_{R,d}^\Lambda(P)$  and  $\Omega_{S,d}^\Lambda(Q)$  that

$$\pi_{\mathbf{c}}(\Omega_{T,d}^\Lambda(M)) \subseteq \Omega_{R,d}^\Lambda(P) \times \Omega_{S,d}^\Lambda(Q)$$

whenever  $R$  and  $S$  are respectively the left most column and its complement in a standard tableau  $T$  whose left-most column is labeled by indices in  $[\mathbf{c}]$ . Conversely suppose  $(P_t)_{n \geq t \geq 0} \times (Q_t)_{n \geq t \geq 0} \in \Omega_{R,d}^\Lambda(P) \times \Omega_{S,d}^\Lambda(Q)$  where  $R \in \text{Tab}_{\mathbf{c}}(1^k)$  and  $S \in \text{Tab}_{\bar{\mathbf{c}}}(\bar{\lambda})$ . Let  $1^{d_n} \supseteq \cdots \supseteq 1^{d_0}$  and  $\bar{\lambda}^{(n)} \supseteq \cdots \supseteq \bar{\lambda}^{(0)}$  denote the corresponding  $\mathbf{c}$ -step and  $\bar{\mathbf{c}}$ -step flags of partitions where  $d_t = c_1 + \cdots + c_t$  and  $d_0 = 0$ . Note that  $R$  and  $S$  will be the left most column and complement in a standard tableau  $T$  whose left-most column is labeled by indices in  $[\mathbf{c}]$  if and only if the number of parts of  $\bar{\lambda}^{(t)}$  is less than or equal to the number of parts of  $1^{d_t}$  whenever  $n \geq t \geq 0$ . The latter condition holds

$$\begin{aligned} &\iff \dim \delta(Q_t) \cap P \leq \dim P_t \text{ whenever } n \geq t \geq 0 \\ &\iff \delta(Q_t) \cap P \subseteq P_t \text{ whenever } n \geq t \geq 0 \\ &\iff \delta(P_t \oplus Q_t) \subseteq P_t \oplus Q_t \text{ whenever } n \geq t \geq 0. \end{aligned}$$

Note that  $Q_t$  must be a  $\mathcal{A}$ -submodule of  $M$  since  $q(Q_t)$  is  $\Lambda$ -submodule of  $M/P$  and because  $Q$  itself is actually a  $\mathcal{A}$ -submodule.

In addition  $\delta(Q_t) \cap P$  contains both  $\alpha(Q_t) \cap P$  and  $\beta(Q_t) \cap P$  whenever  $n \geq t \geq 0$ . To see this just note that we may express  $v \in Q$  as  $v = e_0 v + e_1 v$  and so  $\alpha v = \delta e_0 v$  and  $\beta v = \delta e_1 v$  where  $e_0, e_1$  are the idempotents of  $\Lambda$ . Consequently the third implication listed above holds

$$\iff (P_t \oplus Q_t)_{n \geq t \geq 0} \text{ is a } \Lambda\text{-composition series in } M$$

The proposition now follows from the observation

$$\pi_{\mathbf{c}}(P_t \oplus Q_t)_{n \geq t \geq 0} = (P_t)_{n \geq t \geq 0} \times (Q_t)_{n \geq t \geq 0}$$

□

**Corollary 1.** *Let  $M$  be a shape module of parity  $i$  and shape  $\lambda$  and let  $T$  be any standard tableau of shape  $\lambda$ . Either  $\Omega_{T,d}^\Lambda(M)$  is empty or else its Euler characteristic  $\chi(\Omega_{T,d}^\Lambda(M))$  equals one.*

*Proof.* Let  $\dim M = n$  and assume  $\lambda$  has  $k \geq 0$  parts.

The corollary is clearly valid when  $\lambda = 1^k$  because the partition admits only one standard tableau and the corresponding shape module is uniserial — in which case  $\Omega_{T,d}^\Lambda(M)$  is either empty or else is a point and hence its Euler characteristic equals one. Assume now that the number of columns of  $\lambda$  is  $N > 1$  and hypothesize inductively that the corollary holds for all partitions possessing strictly fewer than  $N$  columns.

Let  $T$  be a standard tableau of shape  $\lambda$  and let  $\mathbf{c}$  be the unique bit string in  $\{0, 1\}_k^n$  corresponding to the content of the 1-st column of  $T$ ; namely  $c_t = 1$  if and only if  $t$  labels a box in the first column of  $T$ .

By proposition 1 above  $\pi_{\mathbf{c}}$  maps  $\Omega_{T,d}^\Lambda(M)$  onto  $\Omega_{R,d}^\Lambda(P) \times \Omega_{S,d}^\Lambda(Q)$  where  $R \in \text{Tab}_{\mathbf{c}}(1^k)$  and  $S \in \text{Tab}_{\overline{\mathbf{c}}}(\overline{\lambda})$  are the tableaux obtained by taking the 1-st column and its complement in  $T$ . As mentioned before  $\Omega_{R,d}^\Lambda(P) \simeq \Omega_{\overline{R},e}^\Lambda(P)$  and  $\Omega_{S,d}^\Lambda(Q) \simeq \Omega_{\overline{S},f}^\Lambda(M/P)$  where  $\overline{R}$  and  $\overline{S}$  are standard tableaux of shapes  $1^k$  and  $\overline{\lambda}$  respectively. Since both  $P$  and  $M/P$  are shape modules whose associated partitions have shapes with strictly less than  $N$  columns then we may inductively conclude that both  $\Omega_{R,d}^\Lambda(P)$  and  $\Omega_{S,d}^\Lambda(Q)$  are either empty or have Euler characteristic one.

Clearly  $\Omega_{T,d}^\Lambda(M)$  is empty if and only if either  $\Omega_{R,d}^\Lambda(P)$  is empty or  $\Omega_{S,d}^\Lambda(Q)$  is empty. Let us suppose that both  $\Omega_{R,d}^\Lambda(P)$  and  $\Omega_{S,d}^\Lambda(Q)$  are non-empty and thus have Euler characteristic equal to one.

By adapting lemmas 3.1.1 and 3.2.2 of [11] it follows that a composition series  $(M_t)_{n \geq t \geq 0}$  inside  $\Omega_{T,d}^\Lambda(M)$  will be a pre-image with respect to  $\pi_{\mathbf{c}}$  of a pair  $(P_t)_{n \geq t \geq 0} \times (Q_t)_{n \geq t \geq 0}$  in  $\Omega_{R,d}^\Lambda(P) \times \Omega_{S,d}^\Lambda(Q)$  if and only if there exists a linear map  $\theta : M \rightarrow P$  satisfying

- $P \subseteq \ker \theta$  and  $[e_i, \theta] = 0$  for  $i = 0, 1$
- $[\delta, \theta](Q_t) \subseteq P_t$  and  $[\delta^*, \theta](Q_t) \subseteq P_t$  whenever  $n \geq t \geq 1$  where  $\delta^* = \alpha^* + \beta^*$

such that  $M_t = P_t \oplus_\theta Q_t$  whenever  $n \geq t \geq 0$  where

$$P_t \oplus_\theta Q_t := P_t \oplus \left\{ \theta(x) + x \mid x \in Q_t \right\}.$$

Note that the commutator identities are necessary and sufficient conditions to insure that  $(P_t \oplus_\theta Q_t)_{n \geq t \geq 0}$  is a  $\Lambda$ -composition series in  $M$ . There is a degree of redundancy in this description of the  $\pi_{\mathbf{c}}$ -preimages since

$$P_t \oplus_\theta Q_t = P_t \oplus_\zeta Q_t$$

if and only if  $(\theta - \zeta)(Q_t) \subseteq P_t$  whenever  $n \geq t \geq 0$ . In this case we declare  $\theta$  and  $\zeta$  to be equivalent and write  $\theta \sim \zeta$ . As the authors of [11] point out in lemma 3.2.2 the kernel and commutator constraints and the equivalence relation  $\sim$  are all linear conditions and thus the fiber of preimages under  $\pi_e$  must be affine.

The map  $\pi_e$  is known to be a morphism of varieties and thus it descends to a morphism between  $\Omega_{T,d}^\Lambda(M)$  and  $\Omega_{R,d}^\Lambda(P) \times \Omega_{S,d}^\Lambda(Q)$ . Moreover the Euler characteristic of any of its fibers is equal to one owing to the fact that each fiber is isomorphic to an affine space. By proposition 7.4.1 in [11] we may conclude in this case that

$$\chi\left(\Omega_{T,d}^\Lambda(M)\right) = \chi\left(\Omega_{R,d}^\Lambda(P) \times \Omega_{S,d}^\Lambda(Q)\right)$$

On the other hand Euler characteristic is multiplicative therefore

$$\begin{aligned} \chi\left(\Omega_{T,d}^\Lambda(M)\right) &= \chi\left(\Omega_{R,d}^\Lambda(P) \times \Omega_{S,d}^\Lambda(Q)\right) \\ &= \chi\left(\Omega_{R,d}^\Lambda(P)\right) \cdot \chi\left(\Omega_{S,d}^\Lambda(Q)\right) \\ &= 1 \end{aligned}$$

□

**Corollary 2.** *Let  $M$  be a shape module of parity  $i$ , shape  $\lambda$ , and dimension  $n$ . Let  $\mathbf{d}$  be a bit string in  $\{0, 1\}^n$  and let  $T$  be a standard tableau of shape  $\lambda$ . Then  $\Omega_{T,d}^\Lambda(M)$  will be non-empty if and only if the  $i$ -parity of  $T$  equals  $\mathbf{d}$ .*

*Proof.* Assume  $\lambda$  has  $k$  parts. The corollary is clearly valid when  $\lambda = 1^k$  because the partition admits only one standard tableau and the corresponding shape module is uniserial. In this case  $\Omega_{T,d}^\Lambda(M)$  will be non-empty if and only if  $d_t$  equals the parity  $t + i$  — which is precisely the  $i$ -parity of the box in row  $t$ . Assume now that the number of columns of  $\lambda$  is  $N > 1$  and hypothesize inductively that the corollary holds for partitions with strictly fewer than  $N$  columns.

Let  $T$  be a standard tableau of shape  $\lambda$  and let  $\mathbf{c}$  be the unique bit string in  $\{0, 1\}_k^n$  corresponding to the content of the 1-st column of  $T$ ; namely  $c_t = 1$  if and only if  $t$  labels a box in the first column of  $T$ . Write  $[\mathbf{c}] = \{r_1, \dots, r_k\}$  and  $[\overline{\mathbf{c}}] = \{s_1, \dots, s_{n-k}\}$ .

By proposition 1 we know  $\Omega_{T,d}^\Lambda(M)$  is non-empty if and only if both  $\Omega_{R,d}^\Lambda(P)$  and  $\Omega_{S,d}^\Lambda(Q)$  are non-empty where  $R \in \text{Tab}_e(1^k)$  and  $S \in \text{Tab}_{\overline{e}}(\overline{\lambda})$  are the tableaux obtained by taking the 1-st column and its complement in  $T$ . On the other hand  $\Omega_{R,d}^\Lambda(P)$  and  $\Omega_{S,d}^\Lambda(Q)$  are non-empty if and only if  $\Omega_{\overline{R},e}^\Lambda(P)$  and  $\Omega_{\overline{S},f}^\Lambda(M/P)$  are respectively non-empty; recall that  $\overline{R}$  and  $\overline{S}$  are the standard tableaux obtained from  $R$  and  $S$  by replacing the entries  $r_t$  and  $s_t$  by  $t$  respectively. Also  $e_t = d_{r_t}$  whenever  $k \geq t \geq 1$  and  $f_t = d_{s_t}$  whenever  $n - k \geq t \geq 1$ .

Since both  $P$  and  $M/P$  are shape modules of parities  $i$  and  $1 - i$  and since their respective shapes  $1^k$  and  $\overline{\lambda}$  have fewer than  $N$  columns we may apply the inductive assumption and conclude that

$$\begin{aligned} \Omega_{\overline{R},e}^\Lambda \text{ non-empty} &\iff e_t \text{ equals } i\text{-parity of box } t \text{ in } \overline{R} \\ \Omega_{\overline{S},f}^\Lambda \text{ non-empty} &\iff f_t \text{ equals } (1 - i)\text{-parity of box } t \text{ in } \overline{S} \end{aligned}$$

equivalently

$$\begin{aligned}\Omega_{R,\mathbf{d}}^\Lambda \text{ non-empty} &\iff d_{r_t} \text{ equals } i\text{-parity of box } r_t \text{ in } R \\ \Omega_{S,\mathbf{d}}^\Lambda \text{ non-empty} &\iff d_{s_t} \text{ equals } i\text{-parity of box } s_t \text{ in } S\end{aligned}$$

Bear in mind that the  $(1-i)$ -parity of a box in  $\overline{S}$  is equal to the  $i$ -parity of the same box in  $S$  owing to the fact that first column in  $S$ , which is situated to the immediate right of the first column in  $T$ , is counted as column 1 not 0. The corollary follows given that  $[1 \dots n] = \{d_{r_t} \mid k \geq t \geq 1\} \sqcup \{d_{s_t} \mid n - k \geq t \geq 1\}$ .  $\square$

**Proof of Theorem 1:** By lemma (1) we know that

$$\chi\left(\mathcal{F}_\mathbf{d}^\Lambda(M)\right) = \sum_{T \in \text{Tab}(\lambda)} \chi\left(\Omega_{T,\mathbf{d}}^\Lambda(M)\right)$$

By Corollary (1) each Euler characteristic  $\chi\left(\Omega_{T,\mathbf{d}}^\Lambda(M)\right)$  contributes either 1 or 0 and Corollary (2) stipulates that only standard tableau  $T$  whose  $i$ -parity equals  $\mathbf{d}$  add a non-zero contribution. Therefore the Euler characteristic  $\chi\left(\mathcal{F}_\mathbf{d}^\Lambda(M)\right)$  equals the number of standard tableaux of shape  $\lambda$  whose  $i$ -parity equals  $\mathbf{d}$ .

Taking into account Proposition (1) from the first section we can furthermore conclude:

**Corollary 3.** *Let  $M$  be a shape module of shape  $\lambda$ , parity  $i$ , and dimension  $n$ . Let  $\mathbf{d}$  be a bit string  $\{0,1\}^n$  which can be expressed as  $\mathbf{i}^{\mathbf{j}}$  for some alternating bit string  $\mathbf{i}$  in  $\{0,1\}^k$  and non-negative integer  $k$ -tuple  $\mathbf{j}$  in  $\mathbb{Z}_{\geq 0}^k$ , then the Euler characteristic  $\chi\left(\mathcal{F}_\mathbf{d}^\Lambda(M)\right)$  equals*

$$j_1! \cdots j_k! \left| \text{Chess}_{\mathbf{j}}^{(i^*)}(\lambda) \right|$$

where  $i^*$  is the parity of  $i + i_1 + 1$ .

**Conjecture 1.** *Let  $\lambda$  be a partition of  $n$ , let  $i = 0, 1$  a choice of parity, and let  $\mathbf{d}$  be a bit string in  $\{0,1\}^n$ . Let  $M$  be a shape module of shape  $\lambda$  and parity  $i$  and let  $T \in \text{Tab}^{(i)}(\lambda : \mathbf{d})$ , then  $\Omega_{\mathbf{d},T}^\Lambda(M)$  is an affine space of dimension  $\text{gr}(T)$ . Alternatively, if  $q$  denotes a power of a prime and  $\mathbb{F}_q$  is a finite field with  $q$  elements then the number of  $\mathbb{F}_q$ -rational points of  $\mathcal{F}_\mathbf{d}^\Lambda(M)$  is*

$$\sum_{T \in \text{Tab}^{(i)}(\lambda; \mathbf{d})} q^{\text{gr}(T)}$$

where, accordinging to definition (5) of section (1),  $\text{gr}(T)$  is the number of grounded  $\mathbf{d}$ -transposition pairs of  $T$ .

4. BLOCK-TOEPLITZ REPRESENTATION OF  $\mathrm{SL}_2(\mathcal{L})$ :

In this section we recount a standard realization of the loop group in terms of infinite block-Toeplitz matrices described in lecture 9 of [13].

Consider the variable  $t$  as a coordinate for the circle subgroup  $S^1$  in  $\mathbb{C}^*$  and let  $\mathcal{H} := L^2(S^1; \mathbb{C}^2)$  be the Hilbert space of all square integrable vector-valued functions  $f : S^1 \rightarrow \mathbb{C}^2$  expressed in the coordinate  $t$ . A loop  $g = (g_{ij})$  in  $\mathrm{SL}_2(\mathcal{L})$  gives rise to a multiplication operator  $T_g : \mathcal{H} \rightarrow \mathcal{H}$  defined by

$$T_g \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Note: the component functions of the result of this matrix multiplication are still square integrable since, over a compact domain such as  $S^1$ , the product of a bounded integrable function (in this case a polynomial function) and a square integrable function remains square integrable.

The map  $T_g$  is clearly linear and so we may write down the matrix representing  $T_g$  with respect to the (ordered) Fourier basis  $\{\psi_n \mid n \in \mathbb{Z}\}$  of  $\mathcal{H}$  where

$$\psi_{2j+i} := t^{-j} \vec{e}_i \text{ for } j \in \mathbb{Z} \text{ and } i = 1, 2.$$

The matrix of  $T_g$  will be the  $\mathbb{Z} \times \mathbb{Z}$  matrix whose  $(M, N)$  entry is

$$\mathrm{Res} \frac{g_{ij}}{t^{n-m+1}}$$

where  $M = 2m + i$  and  $N = 2n + j$  with  $i, j \in \{1, 2\}$  and where Res means residue. Alternatively  $T_g$  can be expressed in block form

$$\begin{pmatrix} \ddots & & & & \\ & a_0 & a_1 & a_2 & \\ & a_{-1} & a_0 & a_1 & \\ & a_{-2} & a_{-1} & a_0 & \\ & & & & \ddots \end{pmatrix}$$

where  $a_k := \left( \mathrm{Res} \frac{g_{ij}}{t^{k+1}} \right)$  for  $k \in \mathbb{Z}$ . The  $2 \times 2$  matrix  $a_k$  is precisely the coefficient matrix of  $t_k$  in the Fourier expansion  $g = \sum_{k \in \mathbb{Z}} a_k t^k$ .

Such a  $\mathbb{Z} \times \mathbb{Z}$  matrix in block form with constant block diagonals will be called a *block Toeplitz matrix*. We will identify  $T_g$  with its matrix and the map  $g \mapsto T_g$  defines an injective homomorphism, denoted  $T$ , from the loop group  $\mathrm{SL}_2(\mathcal{L})$  to the *restricted general linear group*  $\mathrm{GL}_{\mathrm{res}}(\mathbb{C})$  (see [17] chapter 6).



**Definition 8.** Let  $\mu \subseteq \lambda$  be a pair of ordered partitions, let  $i = 0, 1$  be a choice of parity, and set  $N := \max(N_\mu, N_\lambda)$ . For an element  $g$  in the loop group  $\mathrm{SL}_2(\mathcal{L})$  define  $\Delta_{\mu, \lambda}^{(i)}(g)$  to be the determinant of the  $N \times N$  submatrix of  $T_g$  whose row and columns sets are  $\mathrm{set}_{N_\lambda}^{(i)}(\mu)$  and  $\mathrm{set}_{N_\lambda}^{(i)}(\lambda)$  respectively.

**Remark 3.** For each  $g$  the minor  $\Delta_{\mu, \lambda}^{(i)}$  is a polynomial in the matrix entries of  $T_g$  and as such is a regular function over  $\mathrm{SL}_2(\mathcal{L})$ .

**Remark 4.** For a non-negative integer  $n$  let  $E_n^{(i)}$  be a short hand notation for the minor  $\Delta_{\emptyset, \lambda}^{(i)}$  where  $\lambda$  is the partition with  $\lambda_0 = n$  and  $\lambda_k = 0$  whenever  $k > 0$ . Evidently  $E_n^{(i)}$  is the  $(i, n+i)$ -entry of  $T_g$ . In view of the fact that the  $(M, N)$  and  $(M+2, N+2)$  entries of  $T_g$  are equal one observes that the following ***i*-Pieri** rule must hold for  $g \in U_+$ :

$$\Delta_{\emptyset, \lambda}^{(i)} = \det \left( E_{p_{st}}^{(q_s)} \right)$$

where  $N = N_\lambda$  and  $(q_0, \dots, q_N)$  is the alternating bit string in  $\{0, 1\}^N$  starting with  $q_0 = i$  and  $p_{st} = \lambda_{N-s} + t - s$  whenever  $N \geq s, t \geq 0$ . Here we use the convention that  $E_n^{(i)} = 0$  whenever  $n$  is negative.

**Definition 9.** For non-negative integers  $m$  and  $n$  let  $\Delta_{m, n}^{(i)}$  be short hand notation for  $\Delta_{\mu, \lambda}^{(i)}$  where  $\mu$  is the partition with  $\mu_k = m - k$  whenever  $m \geq k$  and  $\mu_k = 0$  whenever  $k > m$  and where  $\lambda$  is the partition with  $\lambda_k = n - k$  whenever  $n \geq k$  and  $\lambda_k = 0$  whenever  $k > n$ .

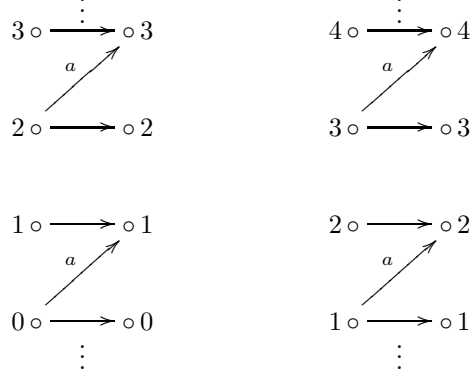
**Remark 5.** Up to re-indexing and a twist as defined in [2] the minors  $\Delta_{0, n}^{(i)}$  are the minors considered in conjecture 4.3 of [2].

**Conjecture 2.** The minors  $\Delta_{m, n}^{(i)}$  are precisely the **generalized minors** of Fomin-Zelevinsky defined in [6].

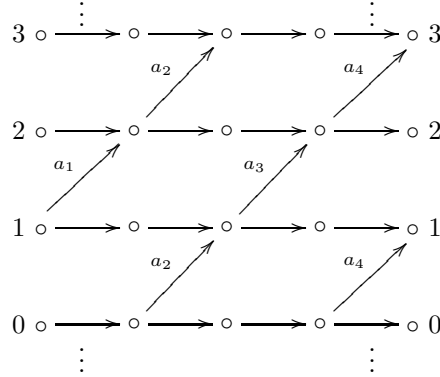
## 5. PATH FORMALISM AND RESOLUTION:

This section describes a locally finite but nevertheless infinite version of the Fomin-Zelevinsky graphical calculus which was originally developed in [7] and [6] as a means to parameterize the double Bruhat cells of a simply connected simple algebraic group.

For  $a \in \mathbb{C}^*$  and a choice of parity  $i = 0, 1$  the *chip diagram*  $\Gamma_i(a)$  is a weighted directed planar graph whose vertices are either sources or sinks: The set of sources and the set of sinks are both indexed by  $\mathbb{Z}$  obeying the rule that *source*  $n$  is connected to sink  $m$  if and only if either  $n = m$  or else  $m = n + 1$  and the parity of  $n$  is  $i$ . Finite portions of the chip diagrams  $\Gamma_0(a)$  and  $\Gamma_1(a)$  are depicted below:



The diagonal edges carry weight  $a$  while all other edges are assumed to carry weight 1. Given an alternating bit string  $\mathbf{i} = (i_1, \dots, i_k)$  in  $\{0, 1\}^k$  and a  $k$ -tuple of parameters  $\mathbf{a} = (a_1, \dots, a_k)$  in  $(\mathbb{C}^*)^k$  let  $\Gamma_{\mathbf{i}}(\mathbf{a})$  denote the graph obtained by *concatenating* the chip diagrams  $\Gamma_{i_1}(a_1), \dots, \Gamma_{i_k}(a_k)$  from left to right starting with  $\Gamma_{i_1}(a_1)$ . To concatenate  $\Gamma_{i_s}(a_s)$  and  $\Gamma_{i_{s+1}}(a_{s+1})$  simply graft each sink of  $\Gamma_{i_s}(a_s)$  with the source of  $\Gamma_{i_{s+1}}(a_{s+1})$  having the same integer label. The set of sources and sinks of the graph  $\Gamma_{\mathbf{i}}(\mathbf{a})$  are both indexed by  $\mathbb{Z}$  which will be respectively depicted on the left and right. For example  $\Gamma_{(1,0,1,0)}(a_1, a_2, a_3, a_4)$  is shown here:



The *weight matrix*  $x(\Gamma_{\mathbf{i}}(\mathbf{a}))$  is the  $\mathbb{Z} \times \mathbb{Z}$  matrix whose  $i, j$  entry is the sum of all weights  $\text{wt}(\pi)$  of paths  $\pi$  joining the  $i$ -th source to the  $j$ -th sink in  $\Gamma_{\mathbf{i}}(\mathbf{a})$ . By definition the weight  $\text{wt}(\pi)$  of a path  $\pi$  is the product of the weights of edges comprising the path  $\pi$ .

Note that the  $\mathbb{Z} \times \mathbb{Z}$  weight matrices  $x(\Gamma_i(a))$  are exactly the block Toeplitz matrices  $T_i(a) := T_{x_i(a)}$  associated with the element  $x_i(a)$  in the loop group  $\text{SL}_2(\mathcal{L})$  for  $i = 0, 1$  and  $a \in \mathbb{C}$ . Weight matrices are multiplicative in the sense that

$$x(\Gamma_{\mathbf{i}}(\mathbf{a})) = x(\Gamma_{i_1}(a_1)) \cdots x(\Gamma_{i_k}(a_k))$$

and therefore

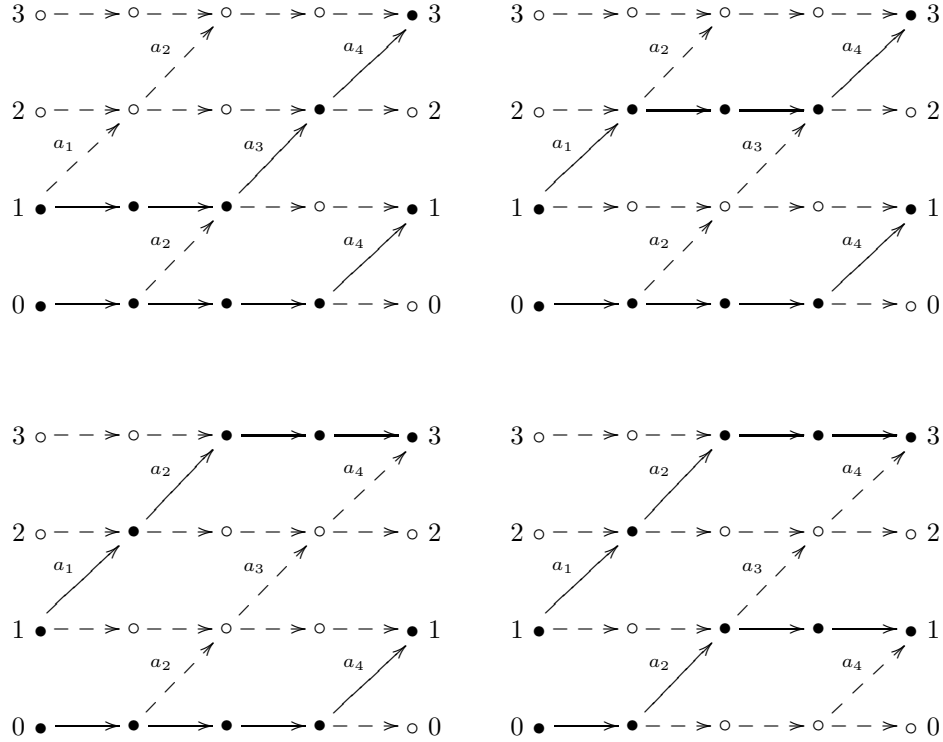
$$(2) \quad x\left(\Gamma_{\mathbf{i}}(\mathbf{a})\right) = T_{i_1}(a_1) \cdots T_{i_k}(a_k).$$

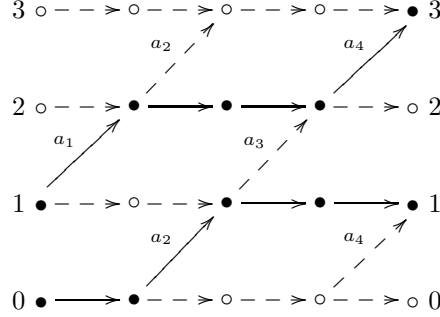
Given two finite subsets  $U = \{u_0, \dots, u_N\}$  and  $V = \{v_0, \dots, v_N\}$  of  $\mathbb{Z}$  the famous *Lindström lemma* (see [7] lemma 1) asserts in the present context that the matrix minor  $\Delta_{U,V}$  of the weight matrix  $x\left(\Gamma_{\mathbf{i}}(\mathbf{a})\right)$  is given by the sum

$$(3) \quad \Delta_{U,V} = \sum_{\substack{\pi = \{\pi_0, \dots, \pi_N\} \\ u_n \xrightarrow{\pi_n} v_n \\ \text{non-crossing}}} \text{wt}(\pi_0) \cdots \text{wt}(\pi_N)$$

where the sum is taken over all families of paths  $\pi = \{\pi_0, \dots, \pi_N\}$  in  $\Gamma_{\mathbf{i}}(\mathbf{a})$  whose members are pairwise non-crossing (i.e. sharing no vertices or edges) and  $\pi_n$  joins source  $u_n$  to sink  $v_n$  whenever  $N \geq n \geq 0$ .

As an illustration consider the case of  $\mathbf{i} = (1, 0, 1, 0)$  with  $U = \{0, 1\}$  and  $V = \{1, 3\}$ . The corresponding families of non-crossing paths  $\pi = \{\pi_0, \pi_1\}$  in  $\Gamma_{\mathbf{i}}(\mathbf{a})$  are depicted below (in bold font):





which agrees with the manual computation of the matrix minor shown here

$$\begin{aligned}
& \Delta_{U,V} \left( x \left( \Gamma_{\mathbf{i}}(\mathbf{a}) \right) \right) \\
&= \Delta_{U,V} \left( x \left( \Gamma_1(a_1) \right) \cdot x \left( \Gamma_0(a_2) \right) \cdot x \left( \Gamma_1(a_3) \right) \cdot x \left( \Gamma_0(a_4) \right) \right) \\
&= \Delta_{U,V} \left( T_1(a_1) \cdot T_0(a_2) \cdot T_1(a_3) \cdot T_0(a_4) \right) \\
&= a_3 a_4^2 + a_1 a_4^2 + a_1 a_2^2 + 2 a_1 a_2 a_4
\end{aligned}$$

**Definition 10.** Let  $\mu \subseteq \lambda$  be a pair of ordered partitions, set  $N = N_\lambda$ , and let  $i = 0, 1$  be a choice of parity. In addition let  $\mathbf{i} = (i_1, \dots, i_k)$  be an alternating bit string in  $\{0, 1\}^k$  and let  $\mathbf{a} = (a_1, \dots, a_k) \in (\mathbb{C}^*)^k$ . Define  $\text{Path}_{\mathbf{i}}^{(i)}(\mu, \lambda)$  to be the collection of all families of non-crossing paths  $\pi = (\pi_1, \dots, \pi_N)$  in  $\Gamma_{\mathbf{i}}(\mathbf{a})$  which join sources in  $U = \text{set}_N^{(i)}(\mu)$  to sinks in  $V = \text{set}_N^{(i)}(\lambda)$ . For  $\mathbf{j} \in \mathbb{Z}_{\geq 0}^k$  let  $\text{Path}_{\mathbf{i}}^{(i)}(\mu, \lambda; \mathbf{j})$  denote those families of non-crossing paths  $\pi = (\pi_1, \dots, \pi_N)$  in  $\text{Path}_{\mathbf{i}}^{(i)}(\mu, \lambda)$  with weight  $\text{wt}(\pi_1) \cdots \text{wt}(\pi_N) = a_1^{j_1} \cdots a_k^{j_k}$ . Clearly

$$\text{Path}_{\mathbf{i}}^{(i)}(\mu, \lambda) = \bigsqcup_{\mathbf{j} \in \mathbb{Z}_{\geq 0}^k} \text{Path}_{\mathbf{i}}^{(i)}(\mu, \lambda; \mathbf{j})$$

If  $\mu = \emptyset$  we shall use the short hand notation  $\text{Path}_{\mathbf{i}}^{(i)}(\lambda)$  and  $\text{Path}_{\mathbf{i}}^{(i)}(\lambda; \mathbf{j})$  instead.

**Remark 6.** In view of equation (2) and the Lindström lemma (3) it follows that

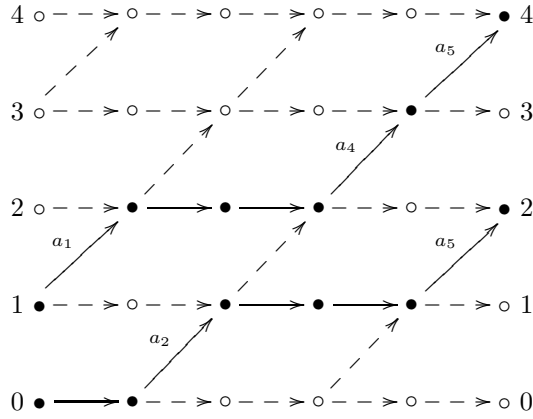
$$\Delta_{\mu, \lambda}^{(i)} \left( x_{i_1}(a_1) \cdots x_{i_k}(a_k) \right) = \sum_{\pi \in \text{Path}_{\mathbf{i}}^{(i)}(\mu, \lambda)} \text{wt}(\pi_0) \cdots \text{wt}(\pi_N)$$

where each  $\pi = \{\pi_0, \dots, \pi_N\}$  is a non-crossing family of paths and where  $\pi_n$  joins source  $u_n = \mu_n + i - n$  to sink  $v_n = \lambda_n + i - n$  whenever  $N \geq n \geq 0$  where  $N = N_\lambda$ .

Consider now the case where  $\mu = \emptyset$ . As before set  $N = N_\lambda$  and let  $u_n = i - n$  and  $v_n = \lambda_n + i - n$  whenever  $N \geq n \geq 0$ . Choose  $\pi = \{\pi_1, \dots, \pi_N\}$  in  $\text{Path}_{\mathbf{i}}^{(i)}(\lambda)$ . An index  $t$  in  $[1 \dots k]$  will be called an *ascent* of a path  $\pi$  in  $\Gamma_{\mathbf{i}}(\mathbf{a})$  if  $\pi$  ascends along

a diagonal edge within the  $\Gamma_{i_t}(\mathbf{a})$  component. Since  $v_n - u_n = \lambda_n$  it must be the case that the path  $\pi_n$  makes exactly  $\lambda_n$  diagonal ascents as it travels from  $u_n$  up to  $v_n$  in  $\Gamma_{\mathbf{i}}(\mathbf{a})$ . Let  $t_{n,1} < \dots < t_{n,\lambda_n}$  be the list of the ascents of  $\pi_n$ . Since the paths are non-crossing it also follows that the position of the  $l$ -th ascent of  $\pi_n$  must be directly under or to the right of the  $l$ -th ascent of  $\pi_{n-1}$ ; equivalently  $t_{n-1,l} \leq t_{n,l}$  whenever  $\lambda_{n-1} \geq l \geq 1$  and whenever  $N \geq n \geq 1$ .

Record (in increasing order and from left to right) the ascents of  $\pi_n$  in the  $n$ -th row of  $\lambda$  and repeat this for each  $n$ . In this way we obtain a semi-standard tableau  $T_\pi$  of shape  $\lambda$  owing to the observations made above. Moreover  $\text{wt}(\pi_1) \cdots \text{wt}(\pi_N) = a_1^{j_1} \cdots a_k^{j_k}$  if and only if the content of  $T_\pi$  is  $\mathbf{j}$ . For example the following pair  $\pi = \{\pi_0, \pi_1\}$  of (highlighted) non-crossing paths



would correspond to the tableau  $T_\pi = \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 5 & \\ \hline \end{array}$

**Proposition 3.** *The map  $\pi \mapsto T_\pi$  is a bijection between  $\text{Path}_{\mathbf{i}}^{(i)}(\lambda)$  and  $\text{Chess}^{(i^*)}(\lambda)$  where  $i^*$  is the parity of  $i + i_1 + 1$ . Moreover  $\pi \in \text{Path}_{\mathbf{i}}^{(i)}(\lambda; \mathbf{j})$  if and only if  $T_\pi \in \text{Chess}_{\mathbf{j}}^{(i^*)}(\lambda)$ .*

*Proof.* Since  $\Gamma_{i_t}(a_t)$  allows ascents only from vertices of parity  $i_t$  and since  $\Gamma_{\mathbf{i}}(\mathbf{a})$  is constructed as an alternating concatenation of such graphs it follows that (1) the parities of the ascents of any path must alternate when read from left to right and (2) the parity of the first ascent equals the parity of  $i + l + 1$  where  $l$  is index of the source vertex. The source vertices of  $\pi_0, \dots, \pi_N$  are  $i, \dots, i - N_\lambda$  respectively. Consequently  $T_\pi \in \text{Chess}^{(i^*)}(\lambda)$ .

Both injectivity and surjectivity follow from the fact that a path in  $\Gamma_{\mathbf{i}}(\mathbf{a})$  is uniquely determined by its source, sink, and list of parity alternating ascents. The refinement of these results to weight and content is obviously true.  $\square$

**Proof of Theorem 2:** Let  $\lambda$  be a partition, let  $i = 0, 1$  be a choice of parity, let  $\mathbf{i} = (i_1, \dots, i_k)$  be an alternating bit string in  $\{0, 1\}^k$ , and let  $\mathbf{a} = (a_1, \dots, a_k)$  be a  $k$ -tuple in  $(\mathbb{C}^*)^k$  then

$$\begin{aligned}
\Delta_{\emptyset, \lambda}^{(i)}(x_{i_1}(a_1) \cdots x_{i_k}(a_k)) &= \sum_{\pi \in \text{Path}_{\mathbf{i}}^{(i)}(\lambda)} \text{wt}(\pi_0) \cdots \text{wt}(\pi_N) \\
&= \sum_{\mathbf{j} \in \mathbb{Z}_{\geq 0}^k} \sum_{\pi \in \text{Path}_{\mathbf{i}}^{(i)}(\lambda; \mathbf{j})} \text{wt}(\pi_0) \cdots \text{wt}(\pi_N) \\
&= \sum_{\mathbf{j} \in \mathbb{Z}_{\geq 0}^k} \left| \text{Chess}_{\mathbf{j}}^{(i^*)}(\lambda) \right| a_1^{j_1} \cdots a_k^{j_k} \\
&= \sum_{\mathbf{j} \in \mathbb{Z}_{\geq 0}^k} \chi\left(\mathcal{F}_{\mathbf{j}}^{\Lambda}(M)\right) \frac{a_1^{j_1} \cdots a_k^{j_k}}{j_1! \cdots j_k!}
\end{aligned}$$

**Remark 7.** *It is worth mentioning here that if in the right hand side of the formula*

$$\Delta_{\emptyset, \lambda}^{(i)}(x_{i_1}(a_1) \cdots x_{i_k}(a_k)) = \sum_{\mathbf{j} \in \mathbb{Z}_{\geq 0}^k} \left| \text{Chess}_{\mathbf{j}}^{(i^*)}(\lambda) \right| a_1^{j_1} \cdots a_k^{j_k}$$

*the chess condition is dropped the resulting sum*

$$\sum_{\mathbf{j} \in \mathbb{Z}_{\geq 0}^k} \left| \text{Tab}_{\mathbf{j}}(\lambda) \right| a_1^{j_1} \cdots a_k^{j_k}$$

*is precisely the definition of the Schur polynomial  $S_{\lambda}$  in the variables  $a_1, \dots, a_k$ ; see [9]. This observation taken together with the remark made in section (4) about the  $i$ -Pieri rule suggest that the block-Toeplitz minors  $\Delta_{\emptyset, \lambda}^{(i)}$  should be viewed as generalized Schur polynomials.*

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